The dynamics of commodity spot, forward, futures prices and convenience yield∗

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Abstract

This paper studies commodity spot, forward, and futures prices under a continuous-time setting. Our model considers a representative firm, which uses an input commodity to produce an output commodity, stores the commodity, and trades forward or futures commodities to hedge. Through the Hamilton-Jacobi-Bellman equation and Feynman-Kac formula, we derive relations between spot, forward, and futures prices. The convenience yield can be interpreted as shadow price of storage, short selling constraints, and limits of risk. We compare our result with the existing models. The optimal production plan and trading strategy for spot commodity and forward are also derived. Keywords: Commodity price, Convenience yield, Forward price, Futures price, Production Mathematics Subject Classification (2010): 91B28, 91B38, 93E20 JEL Classification: G12, G13, Q02

1 Introduction

The relation between commodity spot, forward, and futures price were first studied by Kaldor (1939), Working (1949), and Brennan (1958)[9, 18, 1]. Kaldor (1939)[9] recognized the importance of convenience yield which is the key factors of evaluating commodity spot, forward, and futures price. Kaldor realized that

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there were differences among commodity spot, forward, and expected prices which were constructed by interest rates, carrying cost, risk premium, and the convenience yield. Working (1949)[18] argued that the convenience yield increases as the amount of storage become scarce through examples from wheat markets which is now called as the theory of storage. Brennan (1958)[1] analyzed the theory of storage empirically.

In 1990, Gibson and Schwartz (1990)[7] introduced mean reverting convenience yields with geometric Brownian spot prices and analyzed futures prices on crude oil. The model enriched the earlier paper to include the stochastic behavior of commodity spot prices and convenience yields. Schwartz (1997), Miltersen and Schwartz (1998), and Casassus and Collin-Dufresne (2005)[17, 10, 2] generalized Gibson and Schwartz model. However, in these papers, the convenience yield is exogenous and defined to be the difference of commodity spot, forward, and futures prices.

On the other hand, there were some research studies from economical aspects. Deaton and Laroque (1992)[4] analyzed the commodity price with storage by modeling producer-consumers and risk-neutral inventory holders under equilibrium. Routledge, Seppi, and Spatt (2000)[16] extended Deaton and Laroque’s model and studied the spot and the forward prices including the convenience yield. Casassus et al. (2013)[3] used production rates and utility functions in their model and showed that convenience yield can be expressed as the marginal productivity rate. However, all of these models do not build from the firm’s profit maximization and thus the relation between commodity spot prices, forward, and futures prices were not analyzed concretely. Nakajima (2015)[11] proposed a model which includes the firm’s profit maximization and consumer-speculator’s utility maximization under a discrete-time setting and derived the equation for the commodity spot price, forward price, and futures prices.

In this paper, we analyze the relationship between commodity spot price, forward, and futures prices from the perspective of a firm under continuous-time setting. We model a representative firm of an industry which takes one-input commodity, produce one-output commodity, trade forward and store input commodity which enhances Nakajima’s model (2015)[11]. As the firm faces trading constraints and storage constraints, we can interpret the convenience yield as the shadow prices on these constraints. We use the Hamilton-Jacobi-Bellman equation and the Feynman-Kac formula in order to derive the optimal condition and the spot-forward price relation. This allows us to compare with the other financial stochastic models such as the Gibson-Schwartz model which was not accessible for a discrete-time model in Nakajima (2015)[11]. Also, the result will give new insights on commodity pricing model which does not stem on the utility agent based model and it does not assume convenience yield process exogenously.

In Section 2, we set up a one-input and one-output model with forward trading. Here we introduce other models which consider hedging by futures, and hedging by forward on the output commodity. In Section 3, we provide equations for spot commodity prices, future spot commodity prices, forward prices, and futures prices and discuss their implications. We show correspon-
dences with other existing models such as the Gibson-Schwartz model[7]. We derive the optimal amount of buying and using the commodity and the optimal trading strategy for forward. Furthermore, we show that the existence of a speculator-consumer agent implies that we have a different way of pricing commodity forward. Section 4 concludes.

2 The one-input and one-output model

2.1 A firm

We consider a representative firm which uses commodity 2, e.g. coal or crude oil, to produce commodity 1, e.g. electricity or heating oil. Let \((\Omega,\mathcal{F},\{\mathcal{F}_t\}_{0 \leq t \leq T}, P)\) be a filtered probability space. \(P\) is the risk-neutral probability and the corresponding natural probability is \(P_N\) which will be considered when we are discussing speculators’ behavior. Let \(p: D \rightarrow \mathbb{R}\) be a production function for commodity 1 which uses commodity 2. Here \(\mathbb{R}^+\) is \(\{x | x \geq 0\}\) and \(\mathbb{R}^+_N\) is \(\{(x_n)_{n=1,...,N} | x_n \geq 0\}\). The prices of spot commodities at time \(t\) are \(S_n(t)\), \(n = 1, 2, 3\). We assume that the storage cost depends on a storage price process \(S_3(t)\) which is positive and \(\mathcal{F}_t\)-adapted. These prices satisfy the following stochastic differential equations.

\[
dS_n(t) = S_n(t) [\mu_{S_n}(t) dt + \sigma_{S_n}(t) \cdot dB(t)], \quad 0 \leq t \leq T, n = 1, 2, 3
\]

where \(B(t)\) is a \(D\)-dimensional standard Brownian motion. We assume that \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\) is a natural filtration generated by \(B(t)\) and augmented by \(P\)-null sets in \(\mathcal{F}\). Commodity 2 can be stored and there are forward contracts for commodity 2 which matures at \(T\). The prices of forward commodity at time \(t\) which mature at \(T\) is \(F_2(t,T)\), \(0 \leq t \leq T\). The forward price \(F_2(t,T)\) follows the stochastic differential equation:

\[
dF_2(t,T) = F_2(t,T) [\mu_{F_2}(t,T) dt + \sigma_{F_2}(t,T) \cdot dB(t)], \quad 0 \leq t \leq T
\]

The forward commodity can be physically delivered with spot commodity or by cash settlement at maturity. Therefore, \(F_2(t,T) = S_2(t)\). Let \(R: D_R \rightarrow \mathbb{R}\) be a cost function of storage of physical commodity 2 which depends on the storage price process \(S_3(t)\). We define the domain of \(p\) and \(R\) in the next section. The Heath-Jarrow-Morton type forward interest rate is \(f(t,T)\) which are modeled by the following stochastic differential equations. Here we assume conditions C.1, C.2, and C.3 of Heath, Jarrow, and Morton (1992)[8].

\[
df(t,T) = \mu_f(t,T) dt + \sigma_f(t,T) dB(t)
\]

and the spot interest rate is \(r(t)\) which is \(f(t,t) = r(t)\). Therefore the price of the bank account at time \(t\) is \(P_0(t) = \exp\left(\int_0^t r(u) du\right)\). We also assume that \(\mu_{S_2}(\cdot), \mu_f(\cdot, T), \sigma_{S_2}(\cdot), \sigma_{F_2}(\cdot, T), \sigma(\cdot, T)\) are continuous on \([0,T]\). We denote \(E_t(\cdot)\) as the conditional expectation given \(\mathcal{F}_t\).
We use the following notations. \(q_{S_2,b}(t)\) is the amount of spot commodity 2 bought at time \(t\), \(q_{S_2,u}(t)\) is the amount of spot commodity 2 used at time \(t\), \(q_{F_2,b}(t,T)\) is the amount of future commodity 2 which matures at time \(T\) bought at time \(t\). Note that the amount of purchased future commodity 2 \(q_{F_2,b}(t,T)\) which is matured before time \(t\) is 0, because there is no forward traded after it matures. We use the notation \(q(t) = (q_{S_2,b}(t), q_{S_2,u}(t), q_{F_2,b}(t,T))\) for these amounts and \(q = (q(t))_{0 \leq t \leq T}\).

The amount of stored commodity is
\[
dx_{S_2}(t) = (q_{S_2,b}(t) - q_{S_2,u}(t))dt, \quad 0 \leq t \leq T
\]
and the amount of storage of forward commodity which matures at time \(T\) are
\[
dx_{F_2}(t,T) = q_{F_2,b}(t,T)dt, \quad 0 \leq t \leq T.
\]

In this model, the delivery can be done physically or by cash settlement. The amount of storage of forward commodity which reached maturity \(x_{F_2}(t,t)\) are spot commodities and the firm can use it for production. Although we introduce other models for hedging by futures on input commodities or forward on output commodities, the model with forward on input commodities will be the one we mainly focus on.

We now formulate a stochastic control problem for a firm. Let us define
\[
x(t) = (x_{S_2}(t), x_{F_2}(t,T), r(t), S_1(t), S_2(t), S_3(t), F_2(t,T))
\]
which the stochastic controlled system is
\[
\begin{align*}
dx(t) &= \mu_x(t, x(t), q(t))dt + \sigma_x(t, x(t), q(t))dB(t) \quad (1) \\
x(0) &= x_0
\end{align*}
\]
and
\[
\begin{align*}
\mu_x(t, x(t), q(t)) &= (q_{S_2,b}(t) - q_{S_2,u}(t), q_{F_2,b}(t,T), \\
\mu_f(t,t), (\mu_{S_n}(t)S_n(t))_{n=1,\ldots,3}, \mu_{F_2}(t,T)F_2(t,T))^\top \\
\sigma_x(t, x(t)) &= (0, 0, \sigma_f(t,t), (\sigma_{S_n}(t)S_n(t))_{n=1,\ldots,3}, \sigma_{F_2}(t,T)F_2(t,T))^\top .
\end{align*}
\]

The first two terms are the amount of storage \((x_{S_2}(t), x_{F_2}(t,T))\), followed by interest rate \(r(t)\), commodity spot prices \(S_n(t)\), and forward prices \(F_2(t,T)\). Note that the firm can control the amount of storage but it can not control interest rates and prices.
The firm’s objective is to maximize its profit.

\[
\sup_{q \in \mathcal{Q}} \mathbb{E} \left[ \int_0^T e^{-\int_0^t r(u)du} \left( p(q_{S2,u}(t))S_1(t) - q_{S2,b}(t)S_2(t) \right. \right. \\
- R(x_{S2}(t), S_3(t)))dt - \int_0^T e^{-\int_0^t r(u)du} q_{F_2,b}(t,T)F_2(t,T)dt \\
\left. + e^{-\int_0^T r(u)du} (x_{S2}(T) + x_{F_2}(T,T))S_2(T) \right].
\]  

where

\[
\mathcal{Q} = \{ q : q \text{ is } \mathcal{F}_t\text{-adapted process , } x_{S2}(t) \geq 0, 0 \leq t \leq T, \\
0 \leq q_{S2,u}(t) \leq K_{S2,u}, 0 \leq t \leq T, \\
0 \leq x_{F_2}(t,T) \leq K, 0 \leq t \leq T, \\
L_{S2,b} \leq q_{S2,b}(t) \leq K_{S2,b}, 0 \leq t \leq T, \\
L_{F_2,b} \leq q_{F_2,b}(t,T) \leq K_{F_2,b}, 0 \leq t \leq T \}.
\]

The first two terms are the firm’s profit from its main business. The third term is the storage cost of spot commodity 2. Therefore, these three terms represent sales minus cost, including the storage cost. The fourth term is the cost of purchasing forward contracts on commodity 2. The last term is the income from disposing storage at time $T$. This is a continuous-time version of Nakajima model (2015)[11].

Note that the firm does not short sell spot commodities or forward contracts. The firm has upper limits for the forward contract in order to limit the forward price risk. Forward can be stored without storage costs unless it is matured. Define $q^*$ to be the optimal control and $x^*$ to be the corresponding optimal state process. The economic intuition of this model is that the firm is not just deciding the production planning, but also the trading strategy to maximize its profits while controlling its storage amount.

### 2.2 Using futures for hedging

In this section, we introduce another model which uses futures as a hedging tool. The results are somewhat the same comparing to the first model.

We assume that the futures are continuously resettled and settled by cash and the futures price is a martingale under risk-neutral probability. For continuous resettlement on futures and its prices, see Duffie (2001)[5], Chapter 8, Section C and D. We denote $G_2(t,T)$ as its price at time $t$ which matures at
time $T$. The firm’s profit maximization problem is

$$
\begin{align*}
\sup_{q \in Q} & E \left[ \int_0^T e^{-\int_0^u r(u)du} \left( p(q_{S_2,u}(t))S_1(t) - q_{S_2,b}(t)S_2(t) - R(x_{S_2}(t), S_3(t)))dt 
+ \int_0^T \int_t^T e^{-\int_0^u r(u)du} q_{G_2,b}(t,T)dG_2(s,T)dt 
+ e^{-\int_0^T r(u)du} x_{S_2}(T)S_2(T) \right] \right].
\end{align*}
$$

where

$$
Q = \{ q : q \text{ is } \mathcal{F}_t \text{-adapted process}, x_{S_2}(t) \geq 0, 0 \leq t \leq T, \\
0 \leq q_{S_2,u}(t) \leq K_{S_2,u}, 0 \leq t \leq T, \\
0 \leq x_{G_2}(t,T) \leq K, 0 \leq t \leq T, \\
L_{S_2,b} \leq q_{S_2,b}(t) \leq K_{S_2,b}, 0 \leq t \leq T, \\
L_{G_2,b} \leq q_{G_2,b}(t,T) \leq K_{G_2,b}, 0 \leq t \leq T \},
$$

$$
dx_{S_2}(t) = (q_{S_2,b}(t) - q_{S_2,u}(t))dt, 0 \leq t \leq T.
$$

2.3 Speculators’ utility maximization problem

In this section, we introduce the speculator-consumer agent. We assume $P_N$ to be the natural probability. Suppose there are $J$ agents. The agent $j$ is defined by the utility function $u_j$. Agent $j$ consumes commodity 1 and trades the money market account, zero-coupon bond, and forward commodity contract, but does not trade spot commodities. The price of money market account and zero-coupon bond will be denoted as $P_0(t)$ and $P(t,T)$, respectively. The zero-coupon bond which the agent trades will only be those that have the same maturities with the forward commodity contract. They own some share of the firm and this share is fixed. Therefore a part of the firm’s profit $\theta_{\pi,j}(t)$ at time $t$ will be agent $j$’s income.

Furthermore, we assume that the forward commodity price follows

$$
dF_2(t,T) = F_2(t,T) [\mu_{F_2}(t,T)dt + \sigma_{F_2}(t,T) \cdot dB_{P_N}(t)],
$$

under natural probability $P_N$.

Let $c_{1,j}(t)$ be the amount of consumption of commodity 1 at time $t$. Let $C$ be the space of nonnegative adapted processes in $\bar{R}$ for consumption at time $0 \leq t \leq T$, $C_T$ be the space of nonnegative random variable in $\bar{R}$ for consumption at time $T$, and $\Theta$ be a space of $\{\mathcal{F}(t)\}$-progressively measurable, $\bar{R}^{2#F_2(0,T)+1}$-valued process for trading strategies. Agent $j$ maximizes the following expected
utility.

\[
\sup_{(c_{1,j}, \theta_j) \in A_j} E_{P_N} \left[ \int_0^T u_j(t, c_{1,j}(t)) dt + U(W(T)) \right]
\]

where

\[ A_j(t_0, T) = \left\{ (c_{1,j}(\cdot), C_1, \theta(\cdot)) \in C \times C_T \times \Theta : \right. \]

\[ W_j(t) = W_j(0) + \left( \int_0^t \theta_{P_0,j}(s) dP_0(s) + \int_0^t \theta_{P,j}(s, T) dP(s, T) \right. + \left. \int_0^t \theta_{F_2,j}(s, T) dF_2(s, T) \right) - \int_0^t c_{1,j}(s) S_1(s) ds, \]

\[ c_{1,j}(t) \geq 0, \theta_{F_2,j}(t, t) = 0, 0 \leq t \leq T \right\}, \]

\[ W_j(0) = W_{j,0}, \]

\[ W_j(t) = \theta_{P_0,j}(t) P_0(t) + \theta_{P,j}(t, T) P(t, T) + \theta_{F_2,j}(t, T) F_2(t, T), 0 \leq t \leq T. \]

and \( E_{P_N} \) is the expectation operator under \( P_N \). Notice that the position of forward vanishes for each period which means the speculator clears out at maturity.

3 Implications under the one-input and one-output model

3.1 Spot price, forward prices, futures price, and convenience yield

Now we provide the relation between spot prices, forward prices, futures prices, and convenience yields under the model.

Let \( L^1(\Omega, P) \) be a space of integrable function on \( \Omega \) with respect to the measure \( P \). We assume the following conditions on storage function \( R \) and production function \( p \).

Assumption 3.1. \( R(q, S_3) \) is a convex function of \( q \) and differentiable with respect to \( q \). There exists a constant \( K \) which satisfies \( |R(x, s)| \leq K(1 + |(x, s)|^k) \). and a function \( h_R \in L^1(\Omega, P) \) such that \( |\partial_q R| \leq h_R \) where \( \partial_q R \) denote the partial derivative with respect to the amount of storage. Furthermore, \( R \) is defined on \( D_R = (-\epsilon, \infty) \times \mathbb{R}_+ \) where \( \epsilon > 0 \).
This condition was assumed in Nakajima (2015)[11]. The domain $D_R$ of $R$ is defined in order to calculate the partial derivative at 0. The firm optimizes its profit for only non-negative $q$ since it can not take negative amounts for storage. Therefore, this last assumption is only to calculate the partial derivative at the boundary.

**Assumption 3.2.** $p$ is nondecreasing, concave, and differentiable. Furthermore, $p$ is defined on $D_p = (-\epsilon, \infty)$ where $\epsilon > 0$.

Now we define the value function and its assumption. Define

$$J_\pi(t_0, x; q(\cdot)) = E \left[ \int_{t_0}^{T} e^{-\int_{t_0}^{u} r(u)du} (p(q_{S_2,u}(t))S_1(t) - q_{S_2,b}(t)S_2(t) - R(x_{S_2}(t), S_3(t)))dt \right.$$

$$\left. - \int_{t_0}^{T} e^{-\int_{t_0}^{u} r(u)du} q_{F_2,b}(t, T) F_2(t, T)dt + e^{-\int_{t_0}^{T} r(u)du} x_{S_2}(T)S_2(T) \right].$$

The value function of the optimization problem (2) is

$$V_\pi(t_0, x) = \sup_{q(\cdot) \in \mathcal{Q}(t_0, T)} J(t_0, x; q(\cdot)). \quad (5)$$

$$V_\pi(T, x) = e^{-\int_{t_0}^{T} r(u)du} (x_{S_2} + x_{F_2,T})S_2(T), x \in \mathbb{R}^{5+2#F_2(0,T)} \quad (6)$$

where

$$\mathcal{Q}(t_0, T) = \{q: q \text{ is } F_t\text{-adapted process}, x_{S_2}(t) \geq 0, t_0 \leq t \leq T, \quad 0 \leq q_{S_2,u}(t) \leq K_{S_2,u}, 0 \leq t \leq T, \quad 0 \leq x_{F_2}(t, T) \leq K, 0 \leq t \leq T, \quad L_{S_2,b} \leq q_{S_2,b}(t) \leq K_{S_2,b}, 0 \leq t \leq T, \quad L_{F_2,b} \leq q_{F_2,b}(t, T) \leq K_{F_2,b}, 0 \leq t \leq T \}.$$

and $x = (x_{S_2}, x_{F_2,T}, r, S_1, S_2, S_3, F_2)$.

We also need the following assumption in order to derive the relation between commodity spot and forward prices.

**Assumption 3.3.** $V_\pi(t, x) \in C^{1,3}([0, T] \times \mathbb{R}^7)$ and $\partial_{x} V_\pi$ is a continuous function on $[0, T] \times \mathbb{R}^7$ for each time interval $[0, T]$.

The relation between commodity spot and forward prices is derived in the following proposition.

**Proposition 3.1.** Let Assumptions 3.1, 3.2, and 3.3 hold. Suppose that there exists an optimal solution for the problem (2). Then the spot and forward price
satisfy the following equations.

\[ S_2(t) = E \left[ e^{-\int_t^T r(u) du} S_2(T) \middle| \mathcal{F}_t \right] + \lambda_{S_2}(t) \]  
\[ F_2(t, T) = P(t, T)^{-1} \left( E \left[ e^{-\int_t^T r(u) du} S_2(T) \middle| \mathcal{F}_t \right] + \lambda_{F_2,0}(t, T) \right) \]  
\[ S_2(t) = E \left[ e^{-\int_t^T r(u) du} \mathcal{F}_t \right] F_2(t, T) + \lambda_{F_2}(t, T) \]  
\[ S_2(t) = p'(q^{*}_{S_2,u}(t))S_1(t) + (\lambda_{S_2,q_u,l}(t, x) - \lambda_{S_2,q_u,u}(t, x)) \]

where

\[ \lambda_{S_2}(t) = \lambda_{S_2,x}(t) + \lambda_{S_2,q_u,l}(t) - \lambda_{S_2,q_u,u}(t) \]
\[ \lambda_{F_2,0}(t, T) = -\lambda_{F_2,x}(t, T) + \lambda_{F_2,q_u,l}(t, T) + \lambda_{F_2,q_u,u}(t, T) \]
\[ \lambda_{F_2}(t, T) = \lambda_{S_2}(t) + \lambda_{F_2,0}(t, T) \]
\[ P(t, T) = E \left[ e^{-\int_t^T r(u) du} \middle| \mathcal{F}_t \right] \]

and \( q^{*}_{S_2,b}, q^{*}_{S_2,u}, q^{*}_{F_2,b} \) be the optimal solution and \( x^* \) be the corresponding optimal state process.

**Proof.** See the Appendix. \( \square \)

\( \lambda_{F_2}(t, T) \) is the residual between the commodity spot price and the discounted forward price minus the discounted storage cost. These findings were also indicated in Nakajima (2015). Therefore, it is natural to interpret this \( \lambda_{F_2}(t, T) \) as the convenience yield or in other words the benefit of holding spot commodity. This convenience yield can be decomposed by marginal storage cost, shadow prices of storage, short selling constraints, and limits of risk. It can also be decomposed to the cost part and the yield part. The cost consists of marginal storage cost and shadow prices associated with the upper limit of purchasing spot commodities, the lower limit of purchasing forward, the non-negativity of forward storage. The yield is composed of shadow prices associated with the nonnegativity of storage commodity, the lower limit of purchasing spot commodity, the upper limit of forward storage, the upper limit of purchasing forward.

We can also derive the dynamics of \( \lambda_{F_2}(t, T) \).
Corollary 3.2. Let Assumptions 3.1, 3.2, and 3.3 hold. Suppose that there exists an optimal solution for the problem (2). Then the dynamics of $\lambda_{F_2}(t, T)$ are

\[
d\lambda_{F_2}(t, T) = S_2(t) \left[ \mu_{S_2}(t) dt + \sigma_{S_2}(t) \cdot dB(t) \right] - \mathcal{P}(t, T) F_2(t, T) \left[ \mu_{F_2}(t, T) dt - \sigma_{F_2}(t, T) \cdot \nabla F_2(t, T) dt \right] + \left\{ - r(t) \right. \\
\left. \left. \quad \mathbb{E} \left[ \int_t^T e^{- \int_s^T \mu_u du} \partial_{x_{S_2}} P(x_{S_2, u}(s), S_3(s)) ds \bigg| \mathcal{F}_t \right] \right\} dt \\
\left. + \partial_x E \left[ \int_t^T e^{- \int_s^T \mu_u du} \partial_{x_{S_2}} P(x_{S_2, u}(s), S_3(s)) ds \bigg| \mathcal{F}_t \right] \right. \\
\left. - \sigma(x, x^*(t), u^*(t)) \right\} dB(t).
\]

where

\[
\sigma_{F_2}(t, T) = - \sigma_f(t, T) \\
\mu_{F_2}(t, T) = r(t) - \mu_f(t, T) + 1/2 \sigma_{F_2}(t, T) \gamma(t).
\]

Proof. See the Appendix.

This gives us the dynamics of the Lagrange multiplier $\lambda_{F_2}(t, T)$ which have a correspondence with the dynamics of traditional convenience yield which are assumed in Gibson-Schwartz model or Schwartz model.

We can also derive the relation between commodity spot and futures prices in a similar manner.

Proposition 3.3. Let Assumptions 3.1, 3.2, and 3.3 hold. Suppose that there exists an optimal solution for the problem (3). Then the spot and forward price satisfy the following equations.

\[
S_2(t) = E \left[ e^{- \int_t^T \mu_u du} S_2(T) \bigg| \mathcal{F}_t \right] + \lambda_{S_2, x}(t) + \lambda_{S_2, q_u}(t) - \lambda_{S_2, q_u}(t) \\
- E \left[ \int_t^T e^{- \int_s^T \mu_u du} \partial_{x_{S_2}} P(x_{S_2, u}(s), S_3(s)) ds \bigg| \mathcal{F}_t \right] \\
S_2(t) = P(t, T) G_2(t, T) + \lambda_{G_2}(t, T) \\
- E \left[ \int_t^T e^{- \int_s^T \mu_u du} \partial_{x_{S_2}} P(x_{S_2, u}(s), S_3(s)) ds \bigg| \mathcal{F}_t \right] \\
S_2(t) = p'(q_{S_2, q_u}(t)) S_1(t) + (\lambda_{S_2, q_u}(t, x) - \lambda_{S_2, q_u}(t, x))
\]


where
\[ \lambda_{S_2}(t) = \lambda_{S_2}(t) + \lambda_{S_2,q,u}(t) - \lambda_{S_2,q,u}(t) \]
\[ \lambda_{G_2,0}(t,T) = Cov \left[ e^{-\int_t^T r(u) du}, S_2(T) \Big| \mathcal{F}_t \right] \]
\[ \lambda_{G_2}(t,T) = \lambda_{S_2}(t) + \lambda_{G_2,0}(t,T) \]
and \( q_{S_2,b}^*, q_{S_2,u}^*, q_{F_2,b}^* \) be the optimal solution and \( x^* \) be the corresponding optimal state process.

Proof. The proof is similar to that of Proposition 3.1. \( \square \)

3.2 Comparison with other commodity pricing models

Let us compare the results with the existing models. We will show a correspondence between the convenience yield from the existing models and the optimal Lagrange multipliers. However, since the optimal Lagrange multipliers in our model are endogenous variables and the convenience yield in the existing models such as Schwartz (1997) model[17] are exogenous variables, there can be no equivalence among these models.

3.2.1 The Gibson and Schwartz (1990) model.

If the dynamics of the commodity spot price \( S_2(t) \) and the convenience yield \( \delta_2(t) \) are
\[ dS_2(t) = (r - \delta_2(t))S_2(t)dt + \sigma S_2(t)dB_1(t) \]
\[ d\delta_2(t) = \kappa \delta_2(\alpha \delta_2 - \delta_2(t) - \theta)dt + \sigma \delta_2 dB_2(t) \]
and the interest rate \( r \) is deterministic, then the Gibson-Schwartz (1990)[7] model asserts that
\[ S_2(t) = F_2(t,T)e^{-r(T-t)+\delta_2(t)\kappa^{-1}_2(1-e^{-\kappa_2(T-t)})-A(T-t)} \]
where \( A(T-t) \) is determined by the parameters including volatilities. See Gibson and Schwartz (1990) and Schwartz (1997)[7, 17] for details on \( A(T-t) \). Here we slightly modified the notation of \( A(T-t) \) and removed \(-r(T-t)\) outside. Comparing with equation (9), we have
\[ e^{-r(T-t)}F_2(t,T) - E \left[ \int_t^T e^{-r(s-t)} \partial_{x_{S_2}} R(x_{S_2}^*(s), S_3(s)) ds \mathcal{F}_t \right] + \lambda(t,T) \]
\[ = F_2(t,T)e^{-\delta_2(t)\kappa^{-1}_2(1-e^{-\kappa_2(T-t)})-A(T-t)} \]
Thus, we have a correspondence between the Lagrange multipliers $\lambda_{f_2}(t, T)$ and the convenience yield $\delta_2(t)$ which is

$$\delta_2(t) = \kappa_{\delta_2}(1 - e^{-\kappa_{\delta_2}(T-t)})^{-1} \left\{ A(T - t) + \ln \left( 1 - F_2(t, T)^{-1} \left[ e^{r(T-t)} \lambda(t, T) \right] \right) \right\}.$$  

(13)

Note that $\kappa_{\delta_2}$ and $A(T - t)$ are also part of the Gibson-Schwartz model. Therefore, to be precise, we have a correspondence between the adjusted convenience yield and the optimal Lagrange multipliers.

$$\kappa_{\delta_2}^{-1}(1 - e^{-\kappa_{\delta_2}(T-t)})\delta_2(t) - A(T - t)$$

$$= \ln \left( 1 - F_2(t, T)^{-1} \left[ e^{r(T-t)} \lambda(t, T) \right] \right).$$

Thus, our model is compatible with the Gibson-Schwartz model.

3.2.2 The Schwartz (1997) model.

One of the benchmark models is the Schwartz (1997) model[17]. If the dynamics of the commodity spot price $S_2(t)$ and the convenience yield $\delta_2(t)$ are

$$dS_2(t) = (r - \delta_2(t))S_2(t)dt + \sigma_{S_2}S_2(t)dB_1(t)$$

$$d\delta_2(t) = \kappa_{\delta_2}(\alpha_{\delta_2} - \delta_2(t))dt + \sigma_{\delta_2}dB_2(t)$$

$$dr(t) = \kappa_r(\alpha_r - r(t))dt + \sigma_rdB_2(t)$$

then the Schwartz (1990)[7] model asserts that

$$S_2(t) = G_2(t, T)P(t, T)e^{\delta_2(t)\kappa_{\delta_2}^{-1}(1 - e^{-\kappa_{\delta_2}(T-t)}) - A(T-t)}$$

$$P(t, T) = \exp\left( -\kappa_r^{-1}[1 - e^{\kappa_r(T-t)}] + \kappa_r^{-2}(\kappa_r\alpha_r + \sigma_{S_2, r}) \times \left( (1 - e^{-\kappa_r(T-t)}) - \kappa_r(T - t) - (4\kappa_r^3)^{-1}\sigma_r^2(4(1 - e^{-\kappa_r(T-t)}) - 1) \right) \right)$$

where $A(T - t)$ is determined by the parameters including volatilities. See Schwartz (1997)[17] for details on $A(T - t)$. Here we slightly modified the notation of $A(T - t)$ and removed the interest rate term into the zero-coupon bond price $P(t, T)$ for maturity $T$ outside. Comparing with equation (9), we have

$$P(t, T)G_2(t, T) - E \left[ \int_t^T e^{-\int_s^T r(u)du} \partial_{x_{S_2}} R(x^*_{S_2}(s), S_3(s))ds \mid \mathcal{F}_t \right] + \lambda G_2(t, T)$$

$$= G_2(t, T)P(t, T)e^{\delta_2(t)\kappa_{\delta_2}^{-1}(1 - e^{-\kappa_{\delta_2}(T-t)}) - A(T-t)}$$
Thus, we have a correspondence between the Lagrange multipliers \( \lambda G_2(t, T) \) and the convenience yield \( \delta_2(t) \) which is

\[
\delta_2(t) = \kappa_{\delta_2} (1 - e^{-\kappa_{\delta_2}(T-t)})^{-1} \left[ A(T-t) + \ln \left( 1 - (G_2(t, T)P(t, T))^{-1} \right) \right]
\]

\[
\cdot E \left[ \int_t^T e^{-\int_s^t r(u)du} \partial_x S_2^2 R(x_s^2(s), S_3(s))ds \mid \mathcal{F}_t \right] - \lambda G_2(t, T) \]

Note that \( \kappa_{\delta_2} \) and \( A(T-t) \) are also part of the Gibson-Schwartz model. Therefore, to be precise, we have a correspondence between the adjusted convenience yield and the optimal Lagrange multipliers.

\[
\kappa_{\delta_2}^{-1} (1 - e^{-\kappa_{\delta_2}(T-t)}) \delta_2(t) - A(T-t) = \ln \left( 1 - (G_2(t, T)P(t, T))^{-1} \right) E \left[ \int_t^T e^{-\int_s^t r(u)du} \partial_x S_2^2 R(x_s^2(s), S_3(s))ds \mid \mathcal{F}_t \right] - \lambda G_2(t, T) \]

This implies that our model is compatible with the Schwartz model.

### 3.2.3 The Casassus and Collin-Dufresne (2005) model.

Now, let us examine Casassus and Collin-Dufresne (2005) model[2]. Suppose that the commodity spot price \( S_2(t) \), the convenience yield \( \delta_2(t) \), and the interest rate \( r(t) \) are assumed as follows.

\[
\ln S_2(t) = \phi_0 + \phi^\top Y(t)
\]

\[
r(t) = \psi_0 + \psi_1 Y_1(t)
\]

\[
dY(t) = -\kappa Y Y_1(t) dt + dB_Y(t).
\]

and they derived

\[
\ln G_2(t, T) = A(T - t) + B(T - t)^\top Y(t)
\]

\[
E_t[S_2(T) - S_2(t)] = \int_t^T (r(s) - \delta_2(s)) S_2(s) ds
\]

\[
\delta_2(t) = r(t) - \frac{1}{2} \phi^\top_Y \phi_Y + \phi^\top_Y \kappa Y Y(t).
\]

Let us define

\[
X(t) = (\ln S_2(t), \ln G_2(t, T), r(t))^\top
\]

\[
e(T - t) = (\phi_0, A(T - t), \psi_0)^\top
\]

\[
M(T - t) = (\phi^\top_Y, B(T - t)^\top, (\psi_1, 0, 0)^\top)
\]

If \( M(T - t) \) is invertible, then

\[
Y(t) = M(T - t)^{-1} (X(t) - e(T - t))
\]
Therefore, if we substitute

\[
\ln S_2(t) = \ln \left( E \left[ e^{-\int_t^T r(u)du} |F_t| \right] G_2(t, T) - E \left[ \int_t^T e^{-\int_u^T r(v)dv} \partial_{x_{S_2}} R(x_{S_2}(s), S_3(s))ds |F_t| + \lambda G_2(t, T) \right] \right),
\]

we can calculate the convenience yield \( \delta_2(t) \) through \( X(t) \) and \( Y(t) \). Thus, we have a correspondence between the convenience yield and the optimal Lagrange multiplier.

### 3.3 Optimal production plan and trading strategy

In order to derive the optimal production plan and trading strategy, we need the following assumptions.

**Assumption 3.4.** \( R \) is a strictly convex function of \( x \). \( R \) is essentially smooth on \( x \). Generally, a convex function \( f \) is essentially smooth for \( C = \text{int}(\text{dom} f) \) if \( C \) is not empty, \( f \) is differentiable throughout \( C \), and \( \lim_{n \to \infty} \| \nabla f(x_n) \| = +\infty \) whenever \( x_1, x_2, \ldots \), is a sequence in \( C \) converging to a boundary point \( x \) of \( C \).

**Assumption 3.5.** \( p \) is strictly concave and essentially smooth.

We can derive the optimal amount of spot commodities and forward with these assumptions.

**Proposition 3.4.** Let Assumptions 3.1 – 3.5 hold. Let \( S_1(t) \) be positive. Suppose the problem (2) has an optimal solution for any \( x_0 \). Then the optimal solution is

\[
q_{S_2,u}^*(t) = I_p \left( \frac{S_2(t) - e^{rt} \lambda_{S_2,u}(t)}{S_1(t)} \right)
\]

\[
(x_{S_2}(t), x_{F_2}(t, T)) = I_{R,t}(x_{0,t,T})
\]

where \( I_p \) is the inverse of \( p' \),

\[
I_{R,t}(x_{0,t,T}) = \phi_t^{-1}(x_{0,t,T})
\]

\[
\phi_t(x_{t,T}) = E \left[ \int_t^T e^{-\int_u^T r(v)dv} R(x_{S_2}(s), S_3(s))ds |F_t| \right]
\]

\[
x_{t,T} = (x_{S_2}(t), x_{F_2}(t, T))
\]

\[
x_{0,t,T} = \left( x_{0,t,T, S_2} \right)
\]

\[
x_{0,t,T, S_2} = -S_2(t) + E \left[ e^{-\int_t^T r(u)du} S_2(T) |F_t| + \lambda_{S_2,b}(t) \right]
\]

\[
x_{0,t,T, F_2} = -e^{-\int_t^T r(u)du} F_2(t, T) + E \left[ e^{-\int_t^T r(u)du} S_2(T) |F_t| \right] + \lambda_{F_2,b}(t) - \lambda_{F_3,b}(t, T)
\]
and the optimal trading strategy is
\[ dx_{S_2}(t) = (q_{S_2,b}(t) - q_{S_2,u}(t))dt, \ 0 \leq t \leq T \]
\[ dx_{F_2}(t, T) = q_{F_2,b}(t, T)dt, \ 0 \leq t \leq T. \]

Proof. See the Appendix.

The firm buys \( q^*_{S_2,b}(t) \) and use \( q^*_{S_2,u}(t) \) commodity 2, and trades forward \( q^*_{F_2,b}(t, s) \). \( q^*_{S_2,u}(t) \) is the hedging strategy for the firm. Although the optimal amount used \( q^*_{S_2,u}(t) \) is determined by the two commodity prices, the optimal amount of buying \( q^*_{S_2,b}(t) \) do not depend on \( S_1(t) \) explicitly.

Examples of the production function and the storage cost function are shown in Nakajima (2015)[11].

3.4 The speculator’s valuation of forward prices

We now turn to the result for the speculator. Let us assume the following conditions.

Assumption 3.6. \( u_j \) is strictly concave and differentiable. There exists a function \( h_{u_j} \in L^1(\Omega, P) \) where \( |\partial u_j(t, \cdot) / \partial c_1| \leq h_{u_j}, \ t=0, ..., T, s=t+1, ..., T. \) Furthermore, \( u_j \) is essentially smooth.

Define
\[ J_{u}(t_0, (W_j); (c_{1,j}(\cdot), \theta_j(\cdot))) = E\left[ \int_{t_0}^T u_j(t, c_{1,j}(t))dt + U_j(W_j(T)) \right]. \]

The value function of the optimization problem (4) is
\[ V_{u_j}(t_0, (W_j)) = \sup_{(c_{1,j}(\cdot), \theta_j(\cdot)) \in A_j(t_0, T)} J_{u_j}(t_0, W_j; (c_{1,j}(\cdot), \theta_j(\cdot))). \]  (14)
\[ V_{u_j}(T, (W_j)) = U_j(W_j) \]  (15)

where
\[ A_j(t_0, T) = \left\{ (c_{1,j}(\cdot), \theta_j(\cdot)) \in C \times \Theta : W_j(t) = W_j(0) + \int_{t_0}^T \theta_j(t)dX(t) \right. \]
\[ - \int_{t_0}^T c_{1,j}(t)S_1(t)dt, c_{1,j}(t) \geq 0, \theta_{F_2,j}(t, t) = 0, t_0 \leq t \leq T \left\} \]
\[ \int_{t_0}^T \theta_j(t)dX(t) \]
\[ = \int_{t_0}^T \theta_{P_0,j}(t)dP_0(t) + \int_{t_0}^T \theta_{P_j}(t, T)dP(t, T) + \int_{t_0}^T \theta_{F_2,j}(t, T)dF_2(t, T) \]

We assume the following condition.
Assumption 3.7. \( V_j(t, W) \in C^{1,3}([0,T] \times \mathbb{R}) \) for each time interval \([0,T]\) and \( \partial_t W V_j \) is a continuous function.

The following result is a modification of intertemporal asset pricing theory.

Proposition 3.5. Let Assumption 3.6 and 3.7 hold. Suppose there exists a consumer who faces optimization problem (4) and there exists an optimal solution and assume that the optimal consumption \( c_1^*(t) \) is positive. Furthermore, assume that all the wealth at time \( T \) is consumed, i.e. \( W^*(T) = C_1^*(T)S_1(T) \) and define \( U_{T,j}^*(C) = U_j(W) \) where \( W = C \cdot S(T) \). Then

\[
F_2(t,T) = \mathbb{E}_P \left[ \left. \frac{\partial U_{T,j}(C_1^*(T))/S_1(T)}{\partial u_j(t,c_1^*(t))/S_1(t)} \right| {\mathcal F}_t \right] \tag{16}
\]

Proof. See the Appendix.

\( S_1(t) \) is used as a numeraire price. It is the usual intertemporal price relation. The convenience yield does not explicitly appear in the above result. However, if we compare (10) and (16), we may interpret that convenience yields are included in the marginal utility.

Corollary 3.6. Let Assumptions 3.1 – 3.6 hold. Suppose that there exists an optimal solution for problems (2) and (4). Assume that the optimal consumption \( c_1^*(t) \) and the wealth process \( W^*(t) \) is positive. Then

\[
E_P \left[ \left. \frac{\partial U_{T,j}(C_1^*(T))/S_1(T)}{\partial u_j(t,c_1^*(t))/S_1(t)} \right| {\mathcal F}_t \right] - E \left[ \left. e^{-\int_t^T r(u) \, du} S_2(T) \right| {\mathcal F}_t \right] P(t,T)^{-1} = -E \left[ \left. \int_T^T e^{-\int_s^T r(u) \, du} \partial_{x_2} R(x_{S_2}^*(s), S_3(s)) \, ds \right| {\mathcal F}_t \right] P(t,T)^{-1} - \lambda F_{2,0}(t,T)
\]

This corollary states that the differences in the valuation of forward prices between a risk-neutral entity (a firm) and a risk averse entity (a speculator) consist of the future marginal storage cost plus the convenience yield on forward. Another interpretation is that a part of convenience yield is implicitly included in the intertemporal adjustment term

\[
\frac{\partial U_{T,j}(C_1^*(T))/\partial c_1}{\partial u_j(t,c_1^*(t))/\partial c_1} \cdot \frac{S_2(T)}{S_1(T)/S_1(t)}.
\]

4 Conclusion

In this paper, we modeled a firm which uses an input commodity to produce an output commodity and also trades forward or futures on the input commodity in a continuous-time framework. The firm can also store input commodities by paying storage costs. We compared the result with the Gibson-Schwartz model[7] and other models. Although our model can be compared to existing
models such as the Gibson-Schwartz model \cite{7}, our model does not assume any dynamics of the convenience yield explicitly.

The model implied that the optimal Lagrange multiplier can be deemed as the convenience yield. We also derived the dynamic of the optimal Lagrange multiplier.

Furthermore, we derived the optimal production and trading strategy for spot commodities and forward. We introduced two other models which consider futures and hedging using futures on an output commodity.

If we introduce the speculator, we can see that convenience yields are implicitly included in the intertemporal adjustment term. In other words, the valuation of commodity forward and futures can be done in two aspects, which include a risk-averse agent (speculator-consumer) and a risk-neutral agent (firm).

Appendix A: Proof of proposition 3.1

Let us define

\[ J_\pi(t_0, x; q(\cdot)) = E\left[ \int_{t_0}^{T} e^{-\int_{0}^{t} r(u)du} (p(q_{S_2,u}(t))S_1(t) - q_{S_2,b}(t)S_2(t) - R(x_{S_2}(t), S_3(t)))dt \right. \]

\[ \left. - \int_{t_0}^{T} e^{-\int_{0}^{t} r(u)du} q_{F_2,b}(t, T)F_2(t, T)dt + e^{-\int_{0}^{T} r(u)du} (x_{S_2}(T) + x_{F_2}(T, T))S_2(T) \right] \]

The Hamilton-Jacobi-Bellman equation for (2) is

\[ \partial_t V_\pi(t, x) + \sup_{q \in Q(0, T)} G_T(t, x, q, \partial_x V_\pi(t, x), \partial_{xx} V_\pi(t, x), V_\pi(t, x)) = 0 \]

\[ V_\pi(T, x) = e^{-\int_{0}^{T} r(u)du} (x_{S_2} + x_{F_2, T})S_2(T), x \in \mathbb{R} \]

where \( \partial_x \) and \( \partial_{xx} \) are the partial derivatives and

\[ G_T(t, x, q, V_1, V_2, V) = \frac{1}{2} \text{tr}(V_2 \sigma_x(t, x) \sigma_x(t, x)^\top) + V_1 \cdot \mu_x(t, x, q) \]

\[ + (p(q_{S_2,u}(t)))S_1(t) - q_{S_2,b}(t)S_2(t) - R(x_{S_2}(t), S_3(t))) \]

\[ - E\left[ e^{-\int_{0}^{T} r(u)du} |F_\pi| q_{F_2,b}(t, T)F_2(t, T) - r(t)V, \right] \]

\[ Q(0, T) = \{ q : (q_{S_2,b}(t) - q_{S_2,u}(t))1_{x_{S_2}(t)=0} \geq 0, 0 \leq t \leq T, \]

\[ 0 \leq q_{S_2,u}(t) \leq K_{S_2,u}, 0 \leq t \leq T, \]

\[ q_{F_2,b}(t, T)1_{K_{x_{F_2}}(t, T) \leq 0}, q_{F_2,b}(t, T)1_{x_{F_2}(t, T) \geq 0} \geq 0, 0 \leq t \leq T \]

\[ L_{S_2,b} \leq q_{S_2,b}(t) \leq K_{S_2,b}, 0 \leq t \leq T, \]

\[ L_{F_2,b} \leq q_{F_2,b}(t, T) \leq K_{F_2,b}, 0 \leq t \leq T \}. \]
Here \(1_{\mathcal{S}_2(t)}=0, 1_{K \geq x_{\mathcal{F}}(t,T)}\), and \(1_{x_{\mathcal{F}}(t,T) \geq 0}\) are indicator functions. Note that
\[0 \leq x_{\mathcal{S}_2}(t) \text{ and } 0 \leq x_{\mathcal{F}}(t,T) \leq K, 0 \leq t \leq T\] are equivalent to \((q_{S_2,b}(t) - q_{S_2,u}(t))1_{x_{\mathcal{S}_2}(t)=0} \geq 0\) and \(q_{\mathcal{F},b}(t,T)1_{K \geq x_{\mathcal{F}}(t,T)} \leq 0\), \(q_{\mathcal{F},b}(t,T)1_{x_{\mathcal{F}}(t,T) \geq 0} \geq 0, 0 \leq t \leq T\) which was also used by Presman, Sethi, and Zhang (1995)[14]. For the Hamilton-Jacobi-Bellman equation, see Nisio (2015)[12], Chapter 2, Section 0
\[\text{and Fleming and Soner (2006)[6], Chapter IV, Section IV.4, Theorem 4.1, and Chapter IV, Section IV.3, Remark 3.3.}\]

We solve the following optimization problem
\[
\sup_{\pi \in \mathcal{Q}(0,T)} G_T(t, x, q, \partial_x V_\pi(t, x), \partial_{xx} V_\pi(t, x), V_\pi(t, x))
\]
which the first order condition is
\[
0 = \partial_{x_{\mathcal{S}_2}} V_\pi(t, x) - S_2(t) + \lambda_{S_2,x}(t) + \lambda_{S_2,qb}(t) - \lambda_{S_2,qu}(t) (19)
\]
\[
0 = -\partial_{x_{\mathcal{F}_2}} V_\pi(t, x) + p^*_b(q_{S_2,u}(t))S_1(t) + \lambda_{S_2,qb}(t) - \lambda_{S_2,qu}(t) (20)
\]
\[
0 = \partial_{x_{\mathcal{F}_2}} V_\pi(t, x) - \mathbb{E} \left[ e^{-\int^T_t r(u)du} f_2(t, T) \right] F_2(t, T) + \lambda_{F_2,x}(t, T) - \lambda_{F_2,qu}(t, T) (21)
\]

From the Hamilton-Jacobi-Bellman equation (17), we have
\[
G_T(t, x^*(t), q^*(t), \partial_x V_\pi(t, x^*(t)), \partial_{xx} V_\pi(t, x^*(t)), V_\pi(t, x^*(t))) + \partial_t V_\pi(t, x^*(t)) = 0
\]
\[
\geq G_{T^M}(t, x, q, \partial_x V_\pi(t, x), \partial_{xx} V_\pi(t, x), V_\pi(t, x)) + \partial_t V_\pi(t, x),
\]
where \((x^*(t), q^*(t))\) is the optimal solution. The argument here is a modification of some part of a proof from Yong and Zhou (1999), Chapter 5, Section 4.1, pp.252-253. Since \(V_\pi \in C^{1,1}([0, T] \times \mathbb{R}^{n+2} \times \mathcal{F}_2(0,T))\) and \(\partial_x V_\pi\) being continuous, we have
\[
0 = \partial_x G_T(t, x^*(t), q^*(t), \partial_x V_\pi(t, x^*(t)), \partial_{xx} V_\pi(t, x^*(t)), V_\pi(t, x^*(t))) + \partial_t V_\pi(t, x^*(t)).
\]

By the definition of \(G_T\),
\[
0 = \partial_{xx} V_\pi(t, x^*(t)) + \partial_{xx} V_\pi(t, x^*(t)) \mu_x(t, x^*(t), q^*(t)) + \partial_x \mu_x(t, x^*(t), q^*(t)) \partial_x V_\pi(t, x^*(t))
\]
\[
+ \frac{1}{2} \text{tr} \left( \sigma_x(t, x^*(t), q^*(t)) \right) \partial_{xx} V_\pi(t, x^*(t)) \sigma_x(t, x^*(t), q^*(t)))
\]
\[
+ \sum_{j=1}^d \left( \partial_x \sigma^j_x(t, x^*(t), q^*(t)) \right)^T \left( \partial_{xx} V_\pi(t, x^*(t)) \sigma_x(t, x^*(t), q^*(t)) \right)^j
\]
\[+ V_\pi(t, x^*(t)) \partial_x r(t) + r(t) \partial_x V_\pi(t, x^*(t)) + \partial_x f_{T^M}(t, x^*(t), q^*(t)),
\]
where
\[ f_{T_m}(t, x(t), q(t)) = (p(q_{S_2, u}(t))S_1(t) - q_{S_2, b}(t)S_2(t) - R(x_{S_2}(t), S_3(t))) - E \left[ e^{-\int_t^T r(u)du} \right] q_{S_2, b}(t, T)F_2(t, T), \]
\[ \text{tr}(\sigma_x^T \partial_{xxx} V_\pi \sigma_x) = (\text{tr}(\sigma_x^T (\partial_{xx} V_\pi)^T) \sigma_x), \cdots, \text{tr}(\sigma_x^T (\partial_{x} V_\pi)^n) \sigma_x) \]
and
\[ \partial_x V_\pi = \left( (\partial_x V_\pi)^1, \cdots, (\partial_x V_\pi)^n \right)^T \]

For \( \partial_{x_{S_2}} V_\pi \) and \( \partial_{x_{F_2}, T_m} V_\pi \), we have
\[ \partial_{x_{S_2}} V_\pi(t, x^*(t)) + \partial_{x_{S_2}} V_\pi(t, x^*(t))\mu_x(t, x^*(t), q^*(t)) \]
\[ + \frac{1}{2} \text{tr} (\sigma_x(t, x^*(t), q^*(t))^T \partial_{x_{S_2}} V_\pi(t, x^*(t))\sigma_x(t, x^*(t), q^*(t))) \]
\[ = 0 \]
\[ \partial_{x_{F_2}, T_m} V_\pi(t, x^*(t)) + \partial_{x_{F_2}, T_m} V_\pi(t, x^*(t))\mu_x(t, x^*(t), q^*(t)) \]
\[ + \frac{1}{2} \text{tr} (\sigma_x(t, x^*(t), q^*(t))^T \partial_{x_{F_2}, T_m} V_\pi(t, x^*(t))\sigma_x(t, x^*(t), q^*(t))) \]
\[ = 0 \]

Applying Feynman-Kac formula for (22) and (23), we have
\[ \partial_{x_{S_2}} V_\pi(t, x^*(t)) = E \left[ e^{-\int_t^T r(u)du} \partial_{x_{S_2}} V_\pi(T, x^*(T)) \right] \]
\[ = E \left[ -\int_t^T e^{-\int_t^s r(u)du} \partial_{x_{S_2}} R(x_{S_2}(s), S_3(s))ds \right] \]
\[ = E \left[ -\int_t^T e^{-\int_t^s r(u)du} S_2(T) \right] \]
\[ = E \left[ -\int_t^T e^{-\int_t^s r(u)du} \partial_{x_{S_2}} R(x_{S_2}(s), S_3(s))ds \right] \]
\[ \partial_{x_{F_2}, T} V_\pi(t, x^*(t)) = E \left[ e^{-\int_t^T r(u)du} \partial_{x_{F_2}, T} V_\pi(T, x^*(T)) \right] \]
\[ = E \left[ -\int_t^T e^{-\int_t^s r(u)du} \partial_{x_{S_2}} V_\pi(T, x^*(T)) \right] \]
\[ = E \left[ -\int_t^T e^{-\int_t^s r(u)du} S_2(T) \right] . \]

For Feynman-Kac formula, see Pham (2009) [13], Theorem 1.3.17.
Substituting this into equation (19) and (21), we have

\[ S_2(t) = E \left[ e^{-\int_t^T r(u)du} S_2(T) \right] \mathcal{F}_t + \lambda S_{2,x}(t) + \lambda S_{2,q,b,t}(t) - \lambda S_{2,q,b,u}(t) \]

which also derives

\[ S_2(t) = E \left[ e^{-\int_t^T r(u)du} \right] \mathcal{F}_t - \int_t^T e^{-\int_t^s r(u)du} \partial_{x_S} R(x_s^*(s), S_3(s)) ds \mathcal{F}_t \]

Furthermore, from equation (19) and (20) we have

\[ S_2(t) = p'(q_{S_2,u}^*(t))S_1(t) + (\lambda S_{2,q,b,t}(t, x) - \lambda S_{2,q,b,u}(t, x)) \] (27)

**Appendix B: Proof of corollary 3.2**

From (26) we have

\[ \lambda F_2(t, T) = S_2(t) - P(t, T)F_2(t, T) \]

Differentiating this equation leads to

\[ \frac{d\lambda F_2(t, T)}{dt} = \frac{dS_2(t)}{dt} - F_2(t, T) \frac{dP(t, T)}{dt} - P(t, T) \frac{dF_2(t, T)}{dt} - \frac{dP(t, T)}{dt} \frac{dF_2(t, T)}{dt} \]

\[ + \frac{dE}{dt} \left[ \int_t^T e^{-\int_t^s r(u)du} \partial_{x_S} R(x_s^*(s), S_3(s)) ds \right] \mathcal{F}_t \]
From Heath, Jarrow, and Morton (1992)[8], the stochastic differential equation for \( P(t, T_m) \) is

\[
\begin{align*}
\text{d}P(t, T) &= P(t, T)[(\mu_P(t, T))\text{d}t + \sigma_P(t, T) \cdot \text{dB}(t)]. \\
\sigma_P(t, T) &= -\sigma_f(t, T) \\
\mu_P(t, T) &= r(t) - b(t, T) + \sigma_P(t, T)\gamma(t) \\
&= r(t) - \mu_f(t, T) + 1/2\sigma_P(t, T)^T\sigma_P(t, T) + \sigma_P(t, T)\gamma(t)
\end{align*}
\]

under risk-neutral probability. Here \( \gamma(t) \) is the market price of risk. Furthermore, from equation (24) and (24),

\[
E \left[ \int_t^T e^{-\int_t^u \tau(u)\text{d}u} \partial_{x_{S_2}} R(x_{S_2}^*, s, S_3(s)) \text{d}s \bigg| F_t \right] \\
= -\partial_{x_{S_2}} V_\pi(t, x^*(t)) + \partial_{x_{F_2, T}} V_\pi(t, x^*(t))
\]

which implies

\[
\begin{align*}
\text{d}E \left[ \int_t^T e^{-\int_t^u \tau(u)\text{d}u} \partial_{x_{S_2}} R(x_{S_2}^*, s, S_3(s)) \text{d}s \bigg| F_t \right] &= -\partial_{x_{S_2}} V_\pi(t, x^*(t)) + \text{d}\partial_{x_{F_2, T}} V_\pi(t, x^*(t)).
\end{align*}
\]

Applying Ito's formula to \( \partial_{x_{S_2}} V_\pi \) and \( \partial_{x_{F_2, T}} V_\pi \), we have

\[
\begin{align*}
\text{d}\partial_{x_{S_2}} V_\pi(t, x^*(t)) &= -[-r(t)\partial_{x_{S_2}} V_\pi(t, x^*(t)) + \partial_{x_{S_2}} f_T(t, x^*(t), q^*(t)) ]\text{d}t \\
&+ \partial_{x_{S_2}} V_\pi(t, x^*(t))\sigma_x(t, x^*(t), u^*(t))\text{dB}(t) \\
&+ \partial_{x_{F_2, T}} V_\pi(t, x^*(t))
\end{align*}
\]

Thus, we have

\[
\begin{align*}
\text{d}E \left[ \int_t^T e^{-\int_t^u \tau(u)\text{d}u} \partial_{x_{S_2}} R(x_{S_2}^*, s, S_3(s)) \text{d}s \bigg| F_t \right] &= -\partial_{x_{S_2}} V_\pi(t, x^*(t)) + \text{d}\partial_{x_{F_2, T}} V_\pi(t, x^*(t)) \\
&= [-r(t)\partial_{x_{S_2}} V_\pi(t, x^*(t)) + \partial_{x_{S_2}} f_T(t, x^*(t), q^*(t)) ]\text{d}t \\
&- \partial_{x_{S_2}} V_\pi(t, x^*(t))\sigma_x(t, x^*(t), u^*(t))\text{dB}(t) \\
&- [-r(t)\partial_\pi V_\pi(t, x^*(t)) + \partial_\pi f_T(t, x^*(t), q^*(t)) ]\text{d}t \\
&+ \partial_{x_{F_2, T}} V_\pi(t, x^*(t))\sigma_x(t, x^*(t), u^*(t))\text{dB}(t)
\end{align*}
\]
Substituting (2) and (3) into (1), we have
\[
d\lambda_{F_2}(t, T) = S_2(t) [\mu_{S_2}(t)dt + \sigma_{S_2}(t) \cdot dB(t)] \\
- P(t, T)F_2(t, T)[(\mu_P(t, T) + \mu_{F_2}(t, T) - \sigma_P(t, T)^\top \sigma_{F_2}(t, T))dt \\
+ (\sigma_P(t, T) - \sigma_{F_2}(t, T)) \cdot dB(t)] \\
+ [-r(t)(\partial_{x_{S_2}}V_\pi(t, x^*(t)) - \partial_{x_{F_2}}V_\pi(t, x^*(t))) + \partial_{x_{S_2}}R(x_{S_2}^*(t), S_3(t))]dt \\
+ [-\partial_{x_{S_2}}V_\pi(t, x^*(t))\sigma_x(t, x^*(t), u^*(t)) \\
+ \partial_{x_{F_2}}V_\pi(t, x^*(t))\sigma_x(t, x^*(t), u^*(t))]dB(t) \\
= S_2(t) [\mu_{S_2}(t)dt + \sigma_{S_2}(t) \cdot dB(t)] \\
- P(t, T)F_2(t, T)[(\mu_P(t, T) + \mu_{F_2}(t, T) - \sigma_P(t, T)^\top \sigma_{F_2}(t, T))dt \\
+ (\sigma_P(t, T) - \sigma_{F_2}(t, T)) \cdot dB(t)] \\
+ \left\{-r(t) \left( E \left[ \int_t^T e^{-\int_s^T r(u)du} \partial_{x_{S_2}}R(x_{S_2}^*(s), S_3(s)) ds \left| F_t \right. \right] \right) \right\} \\
\cdot \sigma_x(t, x^*(t), u^*(t)) \right\} dB(t).
\]

**Appendix C: Proof of proposition 3.4**

Here we derive the optimal production plan and trading strategy.

**Lemma Appendix C:.1.** For any \( t \),
\[
E_t \left[ \int_t^T e^{-\int_s^T r(u)du} R(x_{S_2}(s), S_3(s)) ds \right]
\]
is a strictly convex function.

Since the proof is similar to that of Lemma 10.5 in Fleming and Soner (2006)[6], Chapter IV, Section IV.10, we omit the proof.

The next lemma shows the existence of an inverse of gradient of \( \phi_t \) at \( x_{l,T} = (x_{S_2}(t), x_{F_2}(t, T)) \).
\[
\phi_t(x_{l,T}) = E \left[ \int_t^T e^{-\int_s^T r(u)du} R(x(s), S_3(s)) ds \left| F_t \right. \right]
\]
Lemma Appendix C: 2. Let $\partial_q R$ be bounded by an integrable function $h_R$ and $R$ be strictly convex and essentially smooth. There exists an inverse of
\[
\phi_t(x_{t,T}) = \left( \frac{\partial_x}{\partial x_{S_2}}, E \left[ \int_t^T e^{-\int_s^T r(u)du} R(x_{S_3}(s), S_3(s)) \big| F_t \right] \right)
\]
We define the inverse functions $I_{R,t}$.

Proof. $E \left[ \int_t^T e^{-\int_s^T r(u)du} R(x_{S_3}(s), S_3(s))ds \right]$ is a strictly convex function of $x_t$. Also, note that
\[
\partial_x E \left[ \int_t^T e^{-\int_s^T r(u)du} R(x_{S_2}(s), S_3(s)) \big| F_t \right] = \partial_x E \left[ \int_t^T e^{-\int_s^T r(u)du} R(x_{S_2}(s), S_3(s)) \big| F_t \right]
\]
by the chain rule.

We have an inverse for
\[
\phi_t(x_{t,T})
\]
from Theorem 23.5, Corollary 23.5.1, and Corollary 26.3.1 of Rockafellar (1970)[15].

If $p$ is strictly concave and essentially smooth, then there exists an inverse function $I_p$ for $p'(\cdot)$ from Theorem 23.5, Corollary 23.5.1, and Corollary 26.3.1 of Rockafellar(1970)[15]. Therefore, from equation (27 ) we can derive
\[
q_{S_2,u}(t) = I_p \left( \frac{S_2(t) - (\lambda_{S_2, q_{u,t}}(t, x) - \lambda_{S_2, q_{u,u}}(t, x))}{S_1(t)} \right),
\]
(1)

From Lemma Appendix C: 2 and equations (24), (25) we have
\[
(x_{S_2}^*(t), x_{F_2}^*(t, T)) = I_{R,t}(x_{0,t,T})
\]
(2)

where
\[
x_{0,t,T} = \left( \begin{array}{c}
x_{0,t,T, S_2} \\
x_{0,t,T, F_2, T_m}
\end{array} \right)
\]
\[
x_{0,t,T, S_2} = -S_2(t) + E \left[ e^{-\int_t^T r(u)du} S_2(T) \big| F_t \right] + \lambda_{S_2, b}(t),
\]
\[
x_{0,t,T, F_2, T} = -e^{-\int_t^T r(u)du} F_2(t, T) + E \left[ e^{-\int_t^T r(u)du} S_2(T) \big| F_t \right]
+ \lambda_{F_2, b_2}(t, T) - \lambda_{F_2, b_2}(t, T)
\]

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Furthermore, we can calculate the optimal trading strategy by
\[ dx_{S_2}(t) = (q_{S_2,b}(t) - q_{S_2,u}(t))dt, \]
\[ 0 \leq t \leq T \]
\[ derv_{F_2}(t, T) = q_{F_2,b}(t, T)dt, 0 \leq t \leq T. \]

Thus, we derived the optimal production plan and trading strategy.

**Appendix D: Proof of proposition 3.5**

In this section, we derive the necessary condition for problem (4). We omit the index \( j \) and denote \( u_j(\cdot), W_j(\cdot), c_{1,j}(\cdot), \theta_j(\cdot) \) to be \( u(\cdot), W(\cdot), c_1(\cdot), \theta(\cdot) \) for simplicity.

Let us define
\[ J_u(t_0, x; (c(\cdot), \theta(\cdot))) = E\left[ \int_{t_0}^{T} u(t, c_1(t))dt + U(W(T)) \right]. \]

Define the gain process \( G(t) \) as
\[ G(t) = G(t_0) + \int_{t_0}^{t} \theta_{P_0}(t)P_0(t)r(t)dt \]
\[ + \int_{t_0}^{t} \theta_P(s, T)P(s, T)[(\mu_P(s, T))ds + \sigma_P(s, T)^{\top}dB(s)] \]
\[ + \int_{t_0}^{t} \theta_{F_2}(s, T)F_2(s, T)[\mu_{F_2}(s, T)ds + \sigma_{F_2}(s, T)^{\top}dB(s)] \]

The wealth process \( W(t) \) is
\[ W(t) = W_{t_0} - \int_{t_0}^{t} c_1(s)S_1(s)ds + G(t) \]

The speculator’s strategy \( \theta \) finances the net consumption process \( c_1(t)S_1(t) \)
\[ W(t) = \theta_{P_0}(t)P_0(t) + \theta_P(t, T)P(t, T) + \theta_{F_2}(t, T)F_2(t, T) \]

Thus, the wealth process follows the following stochastic differential equation:
\[ dW(t) = dG(t) - c_1(t)S_1(t)dt \]
\[ = W(t)r(t)dt \]
\[ + W(t)w_1P_0(t, T)[(\mu_P(t, T) - r(t))]dt + \sigma_P(t, T)^{\top}dB(t)] \]
\[ + W(t)w_2P(t, T)[(\mu_{F_2}(t, T) - r(t))]dt + \sigma_{F_2}(t, T)^{\top}dB(t)] - c_1(t)S_1(t)dt \]
\[ w(t) = (w_{P_0}(t), w_P(t, T), w_{F_2}(t, T))^\top \]
\[ w_{P_0}(t) = \theta_{P_0}(t)P_0(t)/W(t), \]
\[ w_P(t, T) = \theta_P(t, T)P(t, T)/W(t), \]
\[ w_{F_2}(t, T) = \theta_{F_2}(t, T)F_2(t, T)/W(t) \]

and used the fact that the sum of weights is 1.
\[ 1 = w_{P_0}(t) + w_P(t, T) + w_{F_2}(t, T) \]

The dynamics of the wealth process can be expressed as
\[
dW(t) = \left( W(t)(w(t)^\top(\mu_u(t) - r(t)) + r(t)) - c_1(t)S_1(t) \right) dt \\
+ W(t)w(t)^\top\sigma_u(t)dB(t) \]

where
\[ \mu_u(t) = (\mu_P(t, T), \mu_{F_2}(t, T))^\top \]
\[ \sigma_u(t) = (\sigma_P(t, T), \sigma_{F_2}(t, T))^\top \]

This wealth process is the state process.

The value function of the optimization problem (4) can be written as
\[ V_u(t_0, W) = \sup_{(c_1(\cdot), \theta(\cdot)) \in A(t_0, T)} J_u(t_0, x; (c_1(\cdot), \theta(\cdot))) \] (1)
\[ V_u(T, W) = U(W) \] (2)

where now the control is \( w(\cdot) \) which replaces \( \theta(\cdot) \).

\[ V(t_0, W_{t_0}) = \sup_{(c_1(\cdot), w(\cdot)) \in A(t_0, T)} E \left[ \int_{t_0}^T u(t, c_1(t))dt + U(C_1(T)) \right] \] (3)

where
\[
A(t_0, T) = \left\{ (c_1(\cdot), w(\cdot)) \in C \times \Theta_1 : \right. \\
W(t) = W_{t_0} + \int_{t_0}^t W(s)(w(s)^\top(\mu_u(s) - r(s)) + r(s)) - c_1(s)S_1(s)ds \\
+ \int_{t_0}^t W(s)w(s)^\top\sigma_u(s)dB(s)c_1(t) \geq 0, \theta_{F_2}(t, t) = 0, t_0 \leq t \leq T \right\}
\]
and $\Theta_1$ be a space of $\{F(t)\}$-progressively measurable, $\bar{R}^2$ valued process.

The Hamilton-Jacobi-Bellman equation for (3) is

$$0 = \partial_t V_u(t, W) + \sup_{(c_1, w) \in A} G_{u,T_M}(t, W, (c, w), \partial_W V_u(t, W), \partial_{WW} V_u(t, W), V_u(t, W))$$

$$V_u(T_M, W) = U(W), W \in \mathbb{R}$$

where $\partial_W$ and $\partial_{WW}$ are the partial derivatives and

$$G_{u,T_M}(t, W, (c, w), V_1, V_2) = \frac{1}{2}(V_2 W^2 \sigma_u(t) \sigma_u(t)^\top w) + V_1(W^\top (\mu_u(t) - r(t))) + W r(t) - c_1 S + u(t, c_1),$$

$\forall (t, W, (c, w), p, P) \in [0, T] \times \mathbb{R} \times A \times \mathbb{R} \times \mathbb{R}$.

We have the following optimization problem

$$\sup_{(c, w) \in A} G_{u,T}(t, W, (c, w), \partial_W V_u(t, W), \partial_{WW} V_u(t, W), V_u(t, W))$$

and the first order condition for this problem is

$$0 = \partial_{WW} V_u(t, W(t)) W^2(t) \sigma_u(t) \sigma_u(t)^\top w^*(t)$$

$$+ \partial_W V_u(t, W(t)) W(t) (\mu_u(t) - r(t))$$

$$+ \partial_W V_u(t, W(t)) W(t) 1_3 r(t)$$

$$0 = -\partial_W V_u(t, W(t)) S_1(t) + \partial_c u(t, c^*)$$

where $1_n$ is $n \times 1$ column of 1.

From the Hamilton-Jacobi-Bellman equation (4), we have

$$G_{u,T}(t, W^*(t), (c^*_1(t), w^*(t)), \partial_W V_u(t, W^*(t)), \partial_{WW} V_u(t, W^*(t)), V_u(t, W^*(t)))$$

$$+ \partial_t V_u(t, W^*(t)) = 0$$

$$\geq G_{u,T_M}(t, W, (c, w), \partial_W V_u(t, W), \partial_{WW} V_u(t, W), V_u(t, W)) + \partial_t V_u(t, W).$$

where $(W^*(t), (c^*_1(t), w^*(t)))$ is the optimal solution. Since $V_u \in C^{1,2}([0, T] \times \mathbb{R})$ and $\partial_W V_u$ being continuous, we have

$$0 = \partial_{WW} V_u(t, W(t)) W^2(t) w^*(t)^\top \sigma_u(t) \sigma_u(t)^\top w^*(t)$$

$$+ \partial_t V_u(t, W^*(t)).$$

If we multiply optimal weights to the equation (5) and sum it up, we have

$$\partial_{WW} V_u(t, W(t)) W^2(t) w^*(t)^\top \sigma_u(t) \sigma_u(t)^\top w^*(t)$$

$$= -\partial_W V_u(t, W(t)) W(t) w^*(t)^\top (\mu_u(t) - r(t)) - \partial_W V_u(t, W(t)) W(t) w^*(t)^\top 1_3 r(t)$$

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Therefore, if \( W(t) \neq 0 \),

\[
\partial_{W} V_u(t, W(t)) W(t) w^*(t) \tau \sigma_u(t) \sigma_u(t) ^\top w^*(t) = \quad (7)
\]

\[
= -\partial_{W} V_u(t, W(t)) w^*(t) \tau (\mu_u(t) - r(t)) - \partial_{W} V_u(t, W(t)) r(t)
\]

By the definition of \( G_{u,T} \),

\[
\partial_{W} V_u(t, W^*(t)) + \partial_{W} V_u(t, W^*(t)) (W^*(t) w^*(t) \tau (\mu_u(t) - r(t)) + W^*(t) r(t) - c_1^*(t) S(t))
\]

\[
+ \frac{1}{2} (\partial_{W} W W V_u(t, W^*(t)) W^*(t) w^*(t) \tau \sigma_u(t) \sigma_u(t) ^\top w^*(t))
\]

\[
+ (\partial_{W} W V_u(t, W^*(t)) W^*(t) w^*(t) \tau \sigma_u(t) \sigma_u(t) ^\top w^*(t)) = 0,
\]

From equation (7), we have

\[
\partial_{W} V_u(t, W^*(t)) + \partial_{W} V_u(t, W^*(t)) (W^*(t) w^*(t) \tau (\mu_u(t) - r(t)) + W^*(t) r(t) - c_1^*(t) S(t))
\]

\[
+ \frac{1}{2} (\partial_{W} W W V_u(t, W^*(t)) W^*(t) w^*(t) \tau \sigma_u(t) \sigma_u(t) ^\top w^*(t)) = 0,
\]

and if we multiply it by \( F_2(t) \) and use \( \partial_{\theta_{F_2}} V_u(t, W^*(t)) = \partial_{W} V_u(t, W^*(t)) F_2(t, T) \)

\[
\partial_{\theta_{F_2}} V_u(t, W^*(t)) + \partial_{W} V_u(t, W^*(t)) (W^*(t) w^*(t) \tau (\mu_u(t) - r(t))
\]

\[
+ W^*(t) r(t) - c_1^*(t) S(t))
\]

\[
+ \frac{1}{2} (\partial_{W} W \theta_{F_2} V_u(t, W^*(t)) W^*(t) w^*(t) \tau \sigma_u(t) \sigma_u(t) ^\top w^*(t)) = 0.
\]

Note that \( \partial_{W} V_u(T, W^*(T)) = \partial_{W} U(W^*(T)) = \partial_{C} U(C^*(T) S_1(T))/S_1(T) =
\]

\( \partial_{C} U_T(C^*(T))/S_1(T) \).

Applying Feynman-Kac formula for (10), we have

\[
\partial_{\theta_{F_2}} V_u(t, W^*(t)) = E_{\mathcal{F}_t} \left[ \partial_{\theta_{F_2}} V_u(T, W^*(T)) \right]_{\mathcal{F}_t}
\]

\[
= E_{\mathcal{F}_t} \left[ \partial_{C} U_T(C^*(T))/S_1(T) F_2(t, T) /S_1(T) \right]_{\mathcal{F}_t}
\]

From equation (6),

\[
\partial_{\theta} V_u(t, W(t)) = \partial_{W} V_u(t, W(t)) F_2(t, T) = \partial_{\theta} u(t, c^*) F_2(t, T) / S_1(t)
\]

(11)

Thus,

\[
F_2(t, T) = E_{\mathcal{F}_t} \left[ \partial_{C} U_T(C^*(T))/S_1(T) \right]_{\mathcal{F}_t}
\]

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References


